DAY THIRTEEN

Applications of Derivatives

Learning & Revision for the Day

- Derivatives as the Rate of Change
- Increasing and Decreasing Function
- Tangent and Normal to a Curve
- Angle of Intersection of Two Curves
- Rolle's Theorem
- Lagrange's Mean Value Theorem

Derivatives as the Rate of Change

$\frac{dy}{dx}$ is nothing but the rate of change of y, relative to x. If a variable quantity y is some function of time t i.e. $y = f(t)$, then small change in time Δt have a corresponding change Δy in y. Thus, the average rate of change $= \frac{\Delta y}{\Delta t}$

When limit $\Delta t \to 0$ is applied, the rate of change becomes instantaneous and we get the rate of change with respect to at the instant t, i.e.

$$\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt}$$

$\frac{dy}{dt}$ is positive if y increases as t increase and it is negative if y decrease as t increase.

Increasing and Decreasing Function

- A function f is said to be an increasing function in $]a, b[$, if $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$, $\forall\ x_1, x_2 \in]a, b[$.
- A function f is said to be a decreasing function in $]a, b[$, if $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$, $\forall\ x_1, x_2 \in]a, b[$.
- $f(x)$ is known as **increasing**, if $f'(x) \ge 0$ and **decreasing**, if $f'(x) \le 0$.
- $f(x)$ is known as strictly increasing, if $f'(x) > 0$ and strictly decreasing, if $f'(x) < 0$.
- Let $f(x)$ be a function that is continuous in $[a, b]$ and differentiable in (a, b). Then,
  (i) $f(x)$ is an increasing function in $[a, b]$, if $f'(x) > 0$ in (a, b).
  (ii) $f(x)$ is strictly increasing function in (a, b), if $f'(x) > 0$ in (a, b).

(iii) $f(x)$ is a decreasing function in $[a, b]$, if $f'(x) < 0$ in (a, b).

(iv) $f(x)$ is a strictly decreasing function in $[a, b]$, if $f'(x) < 0$ in (a, b).

Monotonic Function

A function f is said to be monotonic in an interval, if it is either increasing or decreasing in that interval.

Results on Monotonic Function

(i) If $f(x)$ is a strictly increasing function on an interval $[a, b]$, then f^{-1} exists and it is also a strictly increasing function.

(ii) If $f(x)$ is strictly increasing function on an interval $[a, b]$ such that it is continuous, then f^{-1} is continuous on $[f(a), f(b)]$.

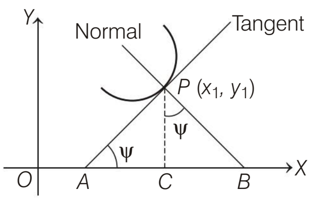
(iii) If $f(x)$ is continuous on $[a, b]$ such that $f'(c) \geq 0$ $[f'(c) > 0]$ for each $c \in (a, b)$, then $f(x)$ is monotonically increasing on $[a, b]$.

(iv) If $f(x)$ is continuous on $[a, b]$ such that $f'(c) \leq 0$ $(f'(c) < 0)$ for each $c \in [a, b]$, then $f(x)$ is monotonically decreasing function on $[a, b]$.

(v) Monotonic function have atmost one root.

Tangent and Normal to a Curve

(i) If a **tangent** is drawn to the curve $y = f(x)$ at a point $P(x_1, y_1)$ and this tangent makes an angle ψ with positive X-direction, then



graph desenber tangents and normal

(a) The slope of the tangent is $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \tan \psi$

(b) Equation of tangent is $y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$

(c) Length of tangent is

$$PA = y_1 \operatorname{cosec} \psi = \left| \frac{y_1 \sqrt{1 + \left(\left(\frac{dy}{dx}\right)_{(x_1, y_1)}\right)^2}}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}} \right|$$

(d) Length of subtangent $AC = y_1 \cot \psi = \left| \frac{y_1}{(dy/dx)_{(x_1, y_1)}} \right|$

(ii) The **normal** to a curve at a point $P(x_1, y_1)$ is a line perpendicular to tangent at P and passing through P, then

(a) The slope of the normal is $-\frac{1}{(dy/dx)_{(x_1, y_1)}}$

(b) Equation of normal is $y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}} (x - x_1)$

(c) Length of normal is $PB = y_1 \sec \psi = \left| y_1 \sqrt{1 + \left(\left(\frac{dy}{dx}\right)_{(x_1, y_1)}\right)^2} \right|$

(d) Length of subnormal is $BC = y_1 \tan \psi = \left| y_1 \left(\frac{dy}{dx}\right)_{(x_1, y_1)} \right|$

Angle of Intersection of Two Curves

The angle of intersection of two curves is defined to be the angle between their tangents, to the two curves at their point of intersection.

The angle between the tangents of the two curves $y = f_1(x)$ and $y = f_2(x)$ is given by

$$\tan \phi = \left| \frac{\left(\frac{dy}{dx}\right)_{\mathrm{I}(x_1, y_1)} - \left(\frac{dy}{dx}\right)_{\mathrm{II}(x_1, y_1)}}{1 + \left(\frac{dy}{dx}\right)_{\mathrm{I}(x_1, y_1)} \left(\frac{dy}{dx}\right)_{\mathrm{II}(x_1, y_1)}} \right| \text{ or } \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

Orthogonal Curves

If the angle of intersection of two curves is a right angle, the two curves are said to intersect orthogonally and the curves are called orthogonal curves.

If $\phi = \frac{\pi}{2}$, $m_1 m_2 = -1 \Rightarrow \left(\frac{dy}{dx}\right)_{\mathrm{I}} \left(\frac{dy}{dx}\right)_{\mathrm{II}} = -1$

NOTE • Two curves touch each other, if $m_1 = m_2$.

Rolle's Theorem

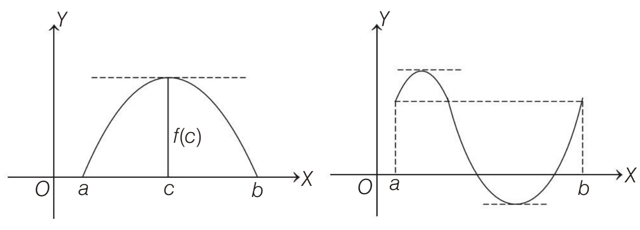
Let f be a real-valued function defined in the closed interval $[a, b]$, such that

(i) $f(x)$ is continuous in the closed interval $[a, b]$.

(ii) $f(x)$ is differentiable in the open interval (a, b).

(iii) $f(a) = f(b)$, then there is some point c in the open interval (a, b), such that $f'(c) = 0$.

Geometrically,



graph of a differentiable function, satisfying the hypothesis of Rolle's theorem.

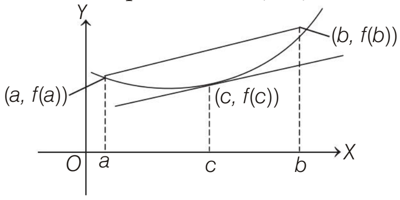
There is atleast one point c between a and b, such that the tangent to the graph at $(c, f(c))$ is parallel to the X-axis.

Algebraic Interpretation of Rolle's Theorem

Between any two roots of a polynomial $f(x)$, there is always a root of its derivative $f'(x)$.

Lagrange's Mean Value Theorem

Let f be a real function, continuous on the closed interval $[a, b]$ and differentiable in the open interval (a, b). Then, there is atleast one point c in the open interval (a, b), such that



graph of a continuous functions explain Lagrange's mean value theorem.

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Geometrically For any chord of the curve $y = f(x)$, there is a point on the graph, where the tangent is parallel to this chord.

Remarks In the particular case, where $f(a) = f(b)$.

The expression $\frac{f(b) - f(a)}{b - a}$ becomes zero.

Thus, when $f(a) = f(b)$, $f'(c) = 0$ for some c in (a, b).

Thus, Rolle's theorem becomes a particular case of the **mean value theorem**.

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 If the volume of a sphere is increasing at a constant rate, then the rate at which its radius is increasing, is

(a) a constant
(b) proportional to the radius
(c) inversely proportional to the radius
(d) inversely proportional to the surface area

2 Moving along the X-axis there are two points with x t^2. The speed with which they are reaching from each other at the time of encounter is (x is in centimetre and t is in seconds)

(a) 16 cm/s (b) 20 cm/s (c) 8 cm/s (d) 12 cm/s

3 An object is moving in the clockwise direction around the unit circle x^2 1. As it passes through the point $\frac{1}{2}, \frac{\sqrt{3}}{2}$, its y-coordinate is decreasing at the rate of 3 units per second. The rate at which the x-coordinate changes at this point is (in unit per second)

(a) 2 (b) $3\sqrt{3}$ (c) $\sqrt{3}$ (d) $2\sqrt{3}$

4 The position of a point in time 't' is given by x bt^2. Its acceleration at time 't' is

(a) b c^2

5 Water is dripping out from a conical funnel of semi-vertical angle $\frac{}{4}$ at the uniform rate of 2 cm^2/s in the surface area, through a tiny hole at the vertex of the bottom. When the slant height of cone is 4 cm, the rate of decrease of the slant height of water, is ➜

(a) $\frac{\sqrt{2}}{4}$ cm/s (b) $\frac{1}{4}$ cm/s
(c) $\frac{1}{\sqrt{2}}$ cm/s (d) None of these

6 A spherical balloon is being inflated at the rate of 35 cc/min. The rate of increase in the surface area (in cm^2/min) of the balloon when its diameter is 14 cm, is ➜ JEE Mains 2013

(a) 10 (b) $\sqrt{10}$ (c) 100 (d) $10\sqrt{10}$

7 Oil is leaking at the rate of 16 cm^3/s from a vertically kept cylindrical drum containing oil. If the radius of the drum is 7 cm and its height is 60 cm. Then, the rate at which the level of the oil is changing when oil level is 18 cm, is

(a) $\frac{6}{4}$ (b) $\frac{-16}{48\pi}$ (c) $\frac{16}{49\pi}$ (d) $\frac{-16}{47\pi}$

8 Two men A and B start with velocities v at the same time from the junction of two roads inclined at 45° to each other. If they travel by different roads, the rate at which they are being separated. ➜ NCERT Exemplar

(a) $\sqrt{2 - \sqrt{2}} \cdot v$ (b) $\sqrt{2 + \sqrt{2}} \cdot v$
(c) $\sqrt{\sqrt{2} - 1} \cdot v$ (d) $\sqrt{2 + \sqrt{2}} \cdot v$

9 The interval in which the function $f(x) = x^{1/x}$ is increasing, is

(a) $(-\infty, e)$ (b) (e, ∞)
(c) $(-\infty, \infty)$ (d) None of these

10 The function $f(x) = \frac{x}{1 + |x|}$ is

(a) strictly increasing
(b) strictly decreasing
(c) neither increasing nor decreasing
(d) not differential at $x = 0$

11 The length of the longest interval, in which the function $3\sin x - 4\sin^3 x$ is increasing, is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{2}$ (d) π

12 An angle θ, $0 < \theta < \frac{\pi}{2}$, which increases twice as fast as its sine, is ➡ NCERT Exemplar

(a) $\frac{\pi}{2}$ (b) $\frac{3\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

13 The value of x for which the polynomial $2x^3 - 9x^2 + 12x + 4$ is a decreasing function of x, is

(a) $-1 < x < 1$ (b) $0 < x < 2$
(c) $x > 3$ (d) $1 < x < 2$

14 If $f(x) = \frac{1}{x+1} - \log(1+x)$, $x > 0$, then f is

(a) an increasing function
(b) a decreasing function
(c) both increasing and decreasing function
(d) None of the above

15 If $f(x) = \sin x - \cos x$, the interval in which function is decreasing in $0 \le x \le 2\pi$, is

(a) $\left[\frac{5\pi}{6}, \frac{3\pi}{4}\right]$ (b) $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$
(c) $\left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$ (d) None of these

16 If $f(x) = -2x^3 + 21x^2 - 60x + 41$, then

(a) $f(x)$ is decreasing in $(-\infty, 1)$
(b) $f(x)$ is decreasing in $(-\infty, 2)$
(c) $f(x)$ is increasing in $(-\infty, 1)$
(d) $f(x)$ is increasing in $(-\infty, 2)$

17 Function $f(x) = \frac{\lambda \sin x + 6\cos x}{2\sin x + 3\cos x}$ is monotonic increasing, if

(a) $\lambda > 1$ (b) $\lambda < 1$ (c) $\lambda < 4$ (d) $\lambda > 4$

18 The sum of intercepts on coordinate axes made by tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$, is

(a) a (b) $2a$
(c) $2\sqrt{a}$ (d) None of these

19 Line joining the points (0, 3) and (5, –2) is a tangent to the curve $y = \frac{ax}{1+x}$, then

(a) $a = 1 \pm \sqrt{3}$ (b) $a = \phi$
(c) $a = -1 \pm \sqrt{3}$ (d) $a = -2 \pm 2\sqrt{3}$

20 The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is parallel to the X-axis, is ➡ AIEEE 2010

(a) $y = 0$ (b) $y = 1$ (c) $y = 2$ (d) $y = 3$

21 The slope of the tangent to the curve $x = 3t^2 + 1$, $y = t^3 - 1$, at $x = 1$ is

(a) 0 (b) $\frac{1}{2}$ (c) ∞ (d) -2

22 Coordinates of a point on the curve $y = x\log x$ at which the normal is parallel to the line $2x - 2y = 3$, are

(a) (0, 0) (b) (e, e)
(c) $(e^2, 2e^2)$ (d) $(e^{-2}, -2e^{-2})$

23 The tangent drawn at the point (0, 1) on the curve $y = e^{2x}$, meets X-axis at the point

(a) $\left(\frac{1}{2}, 0\right)$ (b) $\left(-\frac{1}{2}, 0\right)$ (c) (2, 0) (d) (0, 0)

24 If the normal to the curve $y^2 = 5x - 1$ at the point $(1, -2)$ is of the form $ax - 5y + b = 0$, then a and b are

(a) $4, -14$ (b) $4, 14$ (c) $-4, 14$ (d) $4, 2$

25 The curve $y = ax^3 + bx^2 + cx + 5$ touches the X-axis at $P(-2, 0)$ and cuts the Y-axis at a point Q, where its gradient is 3. Then,

(a) $a = -\frac{1}{2}, b = -\frac{3}{4}$ and $c = 3$
(b) $a = \frac{1}{2}, b = -\frac{3}{4}$ and $c = -3$
(c) $a = \frac{1}{2}, b = -\frac{1}{4}$ and $c = 3$
(d) None of the above

26 The product of the lengths of subtangent and subnormal at any point of a curve is

(a) square of the abscissa (b) square of the ordinate
(c) constant (d) None of these

27 The tangent at $(1, 7)$ to the curve $x^2 = y - 6$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at

(a) (6, 7) (b) (–6, 7) (c) (6, –7) (d) (–6, –7)

28 If the line $ax + by + c = 0$ is normal to curve $xy + 5 = 0$, then

(a) $a + b = 0$ (b) $a > 0$ (c) $a < 0, b < 0$ (d) $a = -2b$

29 The length of subnormal to the curve $y = \frac{x}{1 - x^2}$ at the point having abscissa $\sqrt{2}$ is

(a) $5\sqrt{2}$ (b) $3\sqrt{3}$ (c) $\sqrt{3}$ (d) $3\sqrt{2}$

30 If m is the slope of a tangent to the curve $e^y = 1 + x^2$, then

(a) $|m| \le 1$ (b) $m > -1$ (c) $m > 1$ (d) $|m| > 1$

31 If the curves $y = a^x$ and $y = b^x$ intersects at angle α, then $\tan\alpha$ is equal to

(a) $\frac{a-b}{1+ab}$ (b) $\frac{\log a - \log b}{1 + \log a \log b}$
(c) $\frac{a+b}{1-ab}$ (d) $\frac{\log a + \log b}{1 - \log a \log b}$

32 If the curves $y^2 = 6x$, $9x^2 + by^2 = 16$ intersect each other at right angles, then the value of b is

(a) 6 (b) $\frac{7}{2}$
(c) 4 (d) $\frac{9}{2}$

33 Angle between the tangents to the curve $y = x^2 - 5x + 6$ at the points (2, 0) and (3, 0) is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{6}$
(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

34 If the curves $\frac{x^2}{\alpha}+\frac{y^2}{4}=1$ and $y^3=16x$ intersect at right angles, then the value of α is ➜ JEE Mains 2013

(a) 2 (b) $\frac{4}{3}$ (c) $\frac{1}{2}$ (d) $\frac{3}{4}$

35 $f(x)$ satisfies the conditions of Rolle's theorem in [1, 2] and $f(x)$ is continuous in [1, 2], then $\int_1^2 f'(x)\,dx$ is equal to

(a) 3 (b) 0 (c) 1 (d) 2

36 If the function $f(x)=x^3-6x^2+ax+b$ satisfies Rolle's theorem in the interval [1, 3] and $f'\left(\frac{2\sqrt{3}+1}{\sqrt{3}}\right)=0$, then

(a) $a=-11$ (b) $a=-6$ (c) $a=6$ (d) $a=11$

37 If $f(x)$ satisfies the conditions for Rolle's theorem in [3, 5], then $\int_3^5 f(x)\,dx$ is equal to

(a) 2 (b) -1 (c) 0 (d) $-\frac{4}{3}$

38 A value of C for which the conclusion of mean value theorem holds for the function $f(x)=\log_e x$ on the interval [1, 3] is ➜ AIEEE 2007

(a) $2\log_3 e$ (b) $\frac{1}{2}\log_e 3$

(c) $\log_3 e$ (d) $\log_e 3$

39 The abscissa of the points of the curve $y=x^3$ in the interval $[-2, 2]$, where the slope of the tangents can be obtained by mean value theorem for the interval $[-2, 2]$, are

(a) $\pm\frac{2}{\sqrt{3}}$ (b) $+\sqrt{3}$

(c) $\pm\frac{\sqrt{3}}{2}$ (d) 0

40 In the mean value theorem, $f(b)-f(a)=(b-a)f'(c)$, if $a=4$, $b=9$ and $f(x)=\sqrt{x}$, then the value of c is

(a) 8.00 (b) 5.25 (c) 4.00 (d) 6.25

DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

1 A kite is moving horizontally at a height of 151.5 m. If the speed of kite is 10 m/s, how fast is the string being let out, when the kite is 250 m away from the boy who is flying the kite? The height of boy is 1.5 m. ➜ NCERT Exemplar

(a) 8 m/s (b) 12 m/s (c) 16 m/s (d) 19 m/s

2 The normal to the curve $y(x-2)(x-3)=x+6$ at the point, where the curve intersects the Y-axis passes through the point

(a) $\left(-\frac{1}{2},-\frac{1}{2}\right)$ (b) $\left(\frac{1}{2},\frac{1}{2}\right)$

(c) $\left(\frac{1}{2},-\frac{1}{3}\right)$ (d) $\left(\frac{1}{2},\frac{1}{3}\right)$

3 The values of a for which the function $(a+2)x^3-3ax^2+9ax-1=0$ decreases monotonically throughout for all real x, are

(a) $a<-2$ (b) $a>-2$

(c) $-3<a<0$ (d) $-\infty<a\le-3$

4 If $f(x)=\frac{x}{\sin x}$ and $g(x)=\frac{x}{\tan x}$, where $0<x\le 1$, then in this interval

(a) both $f(x)$ and $g(x)$ are increasing functions

(b) both $f(x)$ and $g(x)$ are decreasing functions

(c) $f(x)$ is an increasing function

(d) $g(x)$ is an increasing function

5 $f(x)=\int_0^x |\log_2[\log_3\{\log_4(\cos t+a)\}]|\,dt$. If $f(x)$ is increasing for all real values of x, then

(a) $a\in(-1,1)$ (b) $a\in(1,5)$

(c) $a\in(1,\infty)$ (d) $a\in(5,\infty)$

6 If $f(x)$ satisfy all the conditions of mean value theorem in $[0, 2]$. If $f(0)=0$ and $|f'(x)|\le\frac{1}{2}$ for all x in $[0, 2]$, then

(a) $f(x)<2$ (b) $|f(x)|\le 1$

(c) $f(x)=2x$ (d) $f(x)=3$ for atleast one x in [0, 2]

7 If $f'(\sin x)<0$ and $f''(\sin x)>0$, $\forall x\in\left(0,\frac{\pi}{2}\right)$ and $g(x)=f(\sin x)+f(\cos x)$, then $g(x)$ is decreasing in

(a) $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$ (b) $\left(0,\frac{\pi}{4}\right)$ (c) $\left(0,\frac{\pi}{2}\right)$ (d) $\left(\frac{\pi}{6},\frac{\pi}{2}\right)$

8 If $f(x)=(x-p)(x-q)(x-r)$, where $p<q<r$, are real numbers, then application of Rolle's theorem on f leads to

(a) $(p+q+r)(pq+qr+rp)=3$

(b) $(p+q+r)^2=3(pq+qr+rp)$

(c) $(p+q+r)^2>3(pq+qr+rp)$

(d) $(p+q+r)^2<3(pq+qr+rp)$

9 If $f(x)$ is a monotonic polynomial of $2m-1$ degree, where $m\in N$, then the equation $[f(x)+f(3x)+f(5x)+\ldots+f(2m-1)x]=2m-1$ has

(a) atleast one real root (b) $2m$ roots

(c) exactly one real root (d) $(2m+1)$ roots

10 A spherical balloon is filled with 4500 π cu m of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cu m/min, then the rate (in m/min) at which the radius of the balloon decreases 49 min after the leakage began is ➜ AIEEE 2012

(a) $\frac{9}{7}$ (b) $\frac{7}{9}$ (c) $\frac{2}{9}$ (d) $\frac{9}{2}$

11 The normal to the curve $x^2 + 2xy - 3y^2 = 0$ at $(1, 1)$

(a) does not meet the curve again ➜ JEE Mains 2015

(b) meets the curve again in the second quadrant

(c) meets the curve again in the third quadrant

(d) meets the curve again in the fourth quadrant

12 If f and g are differentiable functions in $(0, 1)$ satisfying $f(0) = 2 = g(1), g(0) = 0$ and $f(1) = 6$, then for some $c \in]0, 1[$ ➜ JEE Mains 2014

(a) $2f'(c) = g'(c)$ (b) $2f'(c) = 3g'(c)$

(c) $f'(c) = g'(c)$ (d) $f'(c) = 2g'(c)$

13 If $y = f(x)$ is the equation of a parabola which is touched by the line $y = x$ at the point where $x = 1$, then

(a) $2f'(0) = 3f'(1)$ (b) $f'(1) = 1$

(c) $f(0) + f'(1) + f''(1) = 2$ (d) $2f(0) = 1 + f'(0)$

14 Let $a + b = 4, a < 2$ and $g(x)$ be a monotonically increasing function of x. Then,

$$f(x) = \int_0^a g(x)\,dx + \int_0^b g(x)\,dx$$

(a) increases with increase in $(b - a)$

(b) decreases with increase in $(b - a)$

(c) increases with decreases in $(b - a)$

(d) None of the above

15 The angle of intersection of curves, $y = [|\sin x| + |\cos x|]$ and $x^2 + y^2 = 5$, where $[\cdot]$ denotes greatest integral function is

(a) $\frac{\pi}{4}$ (b) $\tan^{-1}\left(\frac{1}{2}\right)$

(c) $\tan^{-1}(2)$ (d) None of these

16 In [0, 1], Lagrange's mean value theorem is not applicable to

(a) $f(x) = \begin{cases} \frac{1}{2} - x, & x < \frac{1}{2} \\ \left(\frac{1}{2} - x\right)^2, & x \geq \frac{1}{2} \end{cases}$ (b) $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

(c) $f(x) = x|x|$ (d) $f(x) = |x|$

ANSWERS

SESSION 1

1 (d)	**2** (c)	**3** (b)	**4** (d)	**5** (a)	**6** (a)	**7** (c)	**8** (a)	**9** (a)	**10** (a)
11 (a)	**12** (d)	**13** (d)	**14** (b)	**15** (d)	**16** (b)	**17** (d)	**18** (a)	**19** (b)	**20** (d)
21 (a)	**22** (d)	**23** (b)	**24** (a)	**25** (a)	**26** (b)	**27** (d)	**28** (c)	**29** (d)	**30** (a)
31 (b)	**32** (d)	**33** (a)	**34** (b)	**35** (b)	**36** (d)	**37** (d)	**38** (a)	**39** (a)	**40** (d)

SESSION 2

1 (a)	**2** (b)	**3** (d)	**4** (c)	**5** (d)	**6** (b)	**7** (b)	**8** (c)	**9** (a)	**10** (c)
11 (d)	**12** (d)	**13** (b)	**14** (a)	**15** (c)	**16** (a)				

Hints and Explanations

SESSION 1

1 Given that, $\frac{dV}{dt} = k$ (say)

$\because \quad V = \frac{4}{3}\pi R^3 \Rightarrow \frac{dV}{dt} = 4\pi R^2 \frac{dR}{dt}$

$\Rightarrow \frac{dR}{dt} = \frac{k}{4\pi R^2}$

Rate of increasing radius is inversely proportional to its surface area.

2 They will encounter, if

$$10 + 6t = 3 + t^2$$

$\Rightarrow \quad t^2 - 6t - 7 = 0 \Rightarrow \quad t = 7$

At $t = 7$ s, moving in a first point

$$v_1 = \frac{d}{dt}(10 + 6t) = 6 \text{ cm/s}$$

At $t = 7$ s, moving in a second point

$v_2 = \frac{d}{dt}(3 + t^2) = 2t = 2 \times 7 = 14$ cm/s

$\therefore$ Resultant velocity

$$= v_2 - v_1 = 14 - 6 = 8 \text{ cm/s}$$

3 The equation of given circle is

$$x^2 + y^2 = 1$$

On differentiating w.r.t. t, we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$\Rightarrow \quad x\frac{dx}{dt} + y\frac{dy}{dt} = 0$

But we have, $x = \frac{1}{2}, y = \frac{\sqrt{3}}{2}$

and $\frac{dy}{dt} = -3$, then $\frac{1}{2}\frac{dx}{dt} + \frac{\sqrt{3}}{2}(-3) = 0$

$\Rightarrow \quad \frac{dx}{dt} = 3\sqrt{3}$

4 Given point is $x = a + bt - ct^2$

Acceleration in x direction and point is $y = at + bt^2 = \frac{d^2x}{dt^2} = -2c$

and point is $y = at + bt^2$ acceleration in y direction

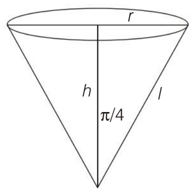
$$= \frac{d^2y}{dt^2} = 2b$$

$\therefore$ Resultant acceleration

$$= \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}$$

$$= \sqrt{(-2c)^2 + (2b)^2} = 2\sqrt{b^2 + c^2}$$

5 If S represents the surface area, then $\frac{dS}{dt} = 2 \text{ cm}^2/\text{s}$



$$S = \pi r l = \pi l \cdot \sin\frac{\pi}{4}\, l = \frac{\pi}{\sqrt{2}} l^2$$

Therefore,

$$\frac{dS}{dt} = \frac{2\pi}{\sqrt{2}}\, l \cdot \frac{dl}{dt} = \sqrt{2}\pi l \cdot \frac{dl}{dt}$$

when $l = 4$ cm, $\frac{dl}{dt} = \frac{2}{\sqrt{2}\pi \cdot 4}$

$$= \frac{1}{2\sqrt{2}\pi} = \frac{\sqrt{2}}{4\pi} \text{ cm / s}$$

6 $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi 3r^2 \frac{dr}{dt}$

$\Rightarrow 35 = 4\pi(7)^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{5}{28\pi}$

Surface area of balloon, $S = 4\pi r^2$

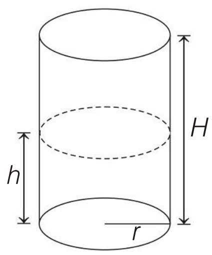
$\therefore \quad \frac{dS}{dt} = 8\pi r \frac{dr}{dt}$

$$= 8\pi \times 7 \times \frac{5}{28\pi} = 10 \text{ cm}^2 / \text{ min}$$

7 Let h be height of oil level at any instant t and V be the volume of oil in cylindrical drum.

Given, $h = 60$ cm, $r = 7$ cm

and $\frac{dV}{dt} = -16 \text{ cm}^3/\text{s}$



$\because \quad V = \pi r^2 h \Rightarrow \frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$

(since, r is constant all the time)

$\Rightarrow -16 = \pi(7)^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = -\frac{16}{49\pi}$

$\Rightarrow \left(\frac{dh}{dt}\right)_{\text{at } h=18} = -\frac{16}{49\pi}$

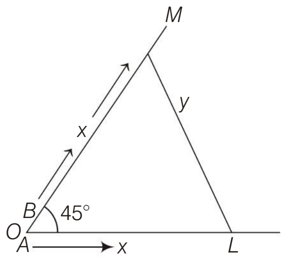
So, height of oil is decreasing at the rate of $\frac{16}{49\pi}$ cm/s.

8 Let L and M be the positions of two men A and B at any time t.

Let $OL = x$ and $LM = y$

Then, $OM = x$

Given, $\frac{dx}{dt} = v$ and we have to find $\frac{dy}{dt}$



From ΔLOM,

$$\cos 45° = \frac{OL^2 + OM^2 - LM^2}{2 \cdot OL \cdot OM}$$

$\Rightarrow \frac{1}{\sqrt{2}} = \frac{x^2 + x^2 - y^2}{2 \cdot x \cdot x} = \frac{2x^2 - y^2}{2x^2}$

$\Rightarrow \sqrt{2}x^2 = 2x^2 - y^2$

$\Rightarrow (2 - \sqrt{2})\, x^2 = y^2$

$\therefore \quad y = \sqrt{2 - \sqrt{2}}\, x$

On differentiating w.r.t. t, we get

$$\frac{dy}{dt} = \sqrt{2 - \sqrt{2}} \frac{dx}{dt}$$

$$= \sqrt{2 - \sqrt{2}}\, v \qquad \left(\because \frac{dx}{dt} = v\right)$$

Hence, they are being separated from each other at the rate $\sqrt{2 - \sqrt{2}}\, v$.

9 Given, $f(x) = x^{1/x}$

$\Rightarrow \quad f'(x) = \frac{1}{x^2}(1 - \log x)\, x^{1/x}$

$f'(x) > 0$, if $1 - \log x > 0$

$\Rightarrow \log x < 1 \Rightarrow x < e$

$\therefore \ f(x)$ is increasing in the interval $(-\infty, e)$.

10 Given, $\quad f(x) = \frac{x}{1 + |x|}$

$$\therefore \ f'(x) = \frac{(1 + |x|) \cdot 1 - x \cdot \frac{|x|}{x}}{(1 + |x|)^2}$$

$$= \frac{1}{(1 + |x|)^2} > 0, \ \forall \ x \in R$$

$\Rightarrow f(x)$ is strictly increasing.

11 Let $f(x) = 3\sin x - 4\sin^3 x = \sin 3x$

Since, $\sin x$ is increasing in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$\therefore \ -\frac{\pi}{2} \le 3x \le \frac{\pi}{2} \Rightarrow -\frac{\pi}{6} \le x \le \frac{\pi}{6}$

Thus, length of interval

$$= \left|\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right| = \frac{\pi}{3}$$

12 Given, $2\frac{d}{dt}(\sin\theta) = \frac{d\theta}{dt}$

$\Rightarrow 2 \times \cos\theta \frac{d\theta}{dt} = \frac{d\theta}{dt}$

$\Rightarrow 2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

13 Let $f(x) = 2x^3 - 9x^2 + 12x + 4$

$\Rightarrow f'(x) = 6x^2 - 18x + 12$

$f'(x) < 0$ for function to be decreasing

$\Rightarrow \ 6(x^2 - 3x + 2) < 0$

$\Rightarrow \ (x^2 - 2x - x + 2) < 0$

$\Rightarrow \quad (x - 2)(x - 1) < 0 \Rightarrow 1 < x < 2$

14 Given curve is $f(x) = \frac{1}{x + 1} - \log(1 + x)$

On differentiating w.r.t. x, we get

$$f'(x) = -\frac{1}{(x+1)^2} - \frac{1}{1 + x}$$

$\Rightarrow f'(x) = -\left[\frac{1}{x + 1} + \frac{1}{(x + 1)^2}\right]$

$\Rightarrow f'(x) = -$ve, when $x > 0$

$\therefore \ f(x)$ is a decreasing function.

15 $\because \ f(x) = \sin x - \cos x$

On differentiating w.r.t. x, we get

$f'(x) = \cos x + \sin x$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right)$$

$$= \sqrt{2}\left(\cos\frac{\pi}{4}\cos x + \sin\frac{\pi}{4}\sin x\right)$$

$$= \sqrt{2}\left[\cos\left(x - \frac{\pi}{4}\right)\right]$$

For decreasing, $f'(x) < 0$

$$\frac{\pi}{2} < \left(x - \frac{\pi}{4}\right) < \frac{3\pi}{2}$$

(within $0 \le x \le 2\pi$)

$\Rightarrow \frac{\pi}{2} + \frac{\pi}{4} < \left(x - \frac{\pi}{4} + \frac{\pi}{4}\right) < \frac{3\pi}{2} + \frac{\pi}{4}$

$\Rightarrow \quad \frac{3\pi}{4} < x < \frac{7\pi}{4}$

16 Given,

$f(x) = -2x^3 + 21x^2 - 60x + 41 \quad$ ...(i)

On differentiating Eq. (i) w.r.t. x, we get

$f'(x) = -6x^2 + 42x - 60$

$= -6(x^2 - 7x + 10)$

$= -6(x - 2)(x - 5)$

If $x < 2$, $f'(x) < 0$ i.e. $f(x)$ is decreasing.

17 $\because \ f(x) = \frac{\lambda\sin x + 6\cos x}{2\sin x + 3\cos x} \quad$ ...(i)

On differentiating w.r.t. x, we get

$$f'(x) = \frac{\begin{bmatrix} (2\sin x + 3\cos x) \\ (\lambda\cos x - 6\sin x) \\ -(\lambda\sin x + 6\cos x) \\ (2\cos x - 3\sin x) \end{bmatrix}}{(2\sin x + 3\cos x)^2}$$

The function is monotonic increasing, if $f'(x) > 0$

$\Rightarrow \quad \frac{3\lambda(\sin^2 x + \cos^2 x) - 12}{(\sin^2 x + \cos^2 x)} > 0$

$\Rightarrow \ 3\lambda - 12 > 0 \qquad (\because \sin^2 x + \cos^2 x = 1)$

$\Rightarrow \quad \lambda > 4$

18 $\sqrt{x} + \sqrt{y} = \sqrt{a}$; (i) $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$

$\therefore \quad \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$

Hence, tangent at (x, y) is

$$Y - y = -\frac{\sqrt{y}}{\sqrt{x}}(X - x)$$

$\Rightarrow \ X\sqrt{y} + Y\sqrt{x} = \sqrt{xy}\,(\sqrt{x} + \sqrt{y})$

$\Rightarrow \ X\sqrt{y} + Y\sqrt{x} = \sqrt{xy}\sqrt{a}$

(using Eq. (i))

$\Rightarrow \quad \dfrac{X}{\sqrt{a}\sqrt{x}} + \dfrac{Y}{\sqrt{a}\sqrt{y}} = 1$

Clearly, its intercepts on the axes are $\sqrt{a}\cdot\sqrt{x}$ and $\sqrt{a}\cdot\sqrt{y}$.

Sum of intercepts $= \sqrt{a}(\sqrt{x}+\sqrt{y})$

$= \sqrt{a}\cdot\sqrt{a} = a$

19 Equation of line joining the points (0, 3) and (5, −2) is $y = 3 - x$. If this line is tangent to $y = \dfrac{ax}{(x+1)}$, then

$(3-x)(x+1) = ax$ should have equal roots.

Thus, $(a-2)^2 + 12 = 0 \Rightarrow$ no value of a

$\Rightarrow a \in \phi$.

20 We have, $y = x + \dfrac{4}{x^2}$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 1 - \frac{8}{x^3}$$

Since, the tangent is parallel to X-axis, therefore

$$\frac{dy}{dx} = 0 \Rightarrow x^3 = 8$$

$\therefore \quad x = 2$ and $y = 3$

21 Given curve is $x = 3t^2 + 1, y = t^3 - 1$

For $x = 1, 3t^2 + 1 = 1 \Rightarrow t = 0$

$\therefore \dfrac{dx}{dt} = 6t, \quad \dfrac{dy}{dt} = 3t^2$

Now, $\quad \dfrac{dy}{dx} = \left(\dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}\right) = \dfrac{3t^2}{6t} = \dfrac{t}{2}$

$\therefore \left(\dfrac{dy}{dx}\right)_{(t=0)} = \dfrac{0}{2} = 0$

22 Given curve is $y = x \log x$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 1 + \log x$$

The slope of the normal

$$= -\frac{1}{(dy/dx)} = \frac{-1}{1+\log x}$$

The slope of the given line $2x - 2y = 3$ is 1.

Since, these lines are parallel.

$\therefore \quad \dfrac{-1}{1+\log x} = 1$

$\Rightarrow \quad \log x = -2$

$\Rightarrow \quad x = e^{-2}$

and $\quad y = -2e^{-2}$

$\therefore$ Coordinates of the point are $(e^{-2}, -2e^{-2})$.

23 Given curve is $y = e^{2x}$

On differentiating w.r.t. x, we get

$\dfrac{dy}{dx} = 2e^{2x} \Rightarrow \left(\dfrac{dy}{dx}\right)_{(0,1)} = 2e^0 = 2$

Equation of tangent at (0, 1) with slope 2 is

$$y - 1 = 2(x - 0) \Rightarrow y = 2x + 1$$

This tangent meets X-axis.

$\therefore \quad y = 0 \Rightarrow 0 = 2x + 1 \Rightarrow x = -\dfrac{1}{2}$

$\therefore$ Coordinates of the point on X-axis is $\left(-\dfrac{1}{2}, 0\right)$.

24 We have, $\quad y^2 = 5x - 1 \quad$ ...(i)

At (1, − 2), $\dfrac{dy}{dx} = \left(\dfrac{5}{2y}\right)_{(1,-2)} = \dfrac{-5}{4}$

$\therefore$ Equation of normal at the point (1, − 2) is

$$[y - (-2)]\left(\frac{-5}{4}\right) + x - 1 = 0$$

$\therefore \quad 4x - 5y - 14 = 0 \quad$ ...(ii)

As the normal is of the form

$$ax - 5y + b = 0$$

On comparing this with Eq. (ii), we get

$$a = 4 \quad \text{and} \quad b = -14$$

25 Since, we have the curve $y = ax^3 + bx^2 + cx + 5$ touches X-axis at $P(-2, 0)$, then X-axis is the tangent at $(-2, 0)$. The curve meets Y-axis in (0, 5).

$$\frac{dy}{dx} = 3ax^2 + 2bx + c$$

$\Rightarrow \left(\dfrac{dy}{dx}\right)_{0,5} = 0 + 0 + c = 3 \quad$ (given)

$\Rightarrow \quad c = 3 \quad$ ...(i)

and $\left(\dfrac{dy}{dx}\right)_{(-2,0)} = 0$

$\Rightarrow \quad 12a - 4b + c = 0 \quad$ [from Eq. (i)]

$\Rightarrow \quad 12a - 4b + 3 = 0 \quad$ ...(ii)

and (−2, 0) lies on the curve, then

$0 = -8a + 4b - 2c + 5$

$\Rightarrow \quad 0 = -8a + 4b - 1 \quad (\because c = 3)$

$\Rightarrow \quad 8a - 4b + 1 = 0 \quad$ ...(iii)

From Eqs. (ii) and (iii), we get

$$a = -\frac{1}{2}, b = -\frac{3}{4}$$

26 Length of subtangent $= y\dfrac{dx}{dy}$

and length of subnormal $= y\dfrac{dy}{dx}$

$\therefore \quad$ Product $= y^2$

$\Rightarrow$ Required product is the square of the ordinate.

27 The tangent to the parabola $x^2 = y - 6$ at (1, 7) is

$x(1) = \dfrac{1}{2}(y + 7) - 6 \Rightarrow y = 2x + 5$

which is also a tangent to the given circle.

i.e. $x^2 + (2x+5)^2 + 16x + 12(2x+5) + c = 0$

$\Rightarrow (5x^2 + 60x + 85 + c = 0)$ must have equal roots.

Let the roots be $\alpha = \beta$.

$\therefore \quad \alpha + \beta = -\dfrac{60}{5}$

$\Rightarrow \quad \alpha = -6$

$\therefore \quad x = -6$ and

$y = 2x + 5 = -7$

28 Given curve, is $xy = -5 < 0$

$\Rightarrow \quad x\dfrac{dy}{dx} + y = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{-y}{x} > 0$

[as $xy = -5 < 0$]

Slope of normal $= \dfrac{-1}{\dfrac{dy}{dx}} = \dfrac{x}{y} < 0$

[as $xy = -5 < 0$]

Hence, slope of normal will be negative.

The line $ax + by + c = 0$

$\Rightarrow \quad by = -ax - c$

$\Rightarrow \quad y = \dfrac{-a}{b}x - \dfrac{c}{b}$

Slope of normal $\dfrac{-a}{b}$ is negative.

$\Rightarrow \quad \dfrac{-a}{b} < 0 \Rightarrow \dfrac{a}{b} > 0$

$\Rightarrow a > 0, b > 0 \quad$ or $\quad a < 0, b < 0$

29 Given, $y = \dfrac{x}{1-x^2}$

At $\quad x = \sqrt{2}, y = -\sqrt{2}$, therefore point is $(\sqrt{2}, -\sqrt{2})$.

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1+x^2}{(1-x^2)^2}$

$\Rightarrow \left(\dfrac{dy}{dx}\right)_{(\sqrt{2}, -\sqrt{2})} = \dfrac{1+2}{(1-2)^2} = 3$

$\therefore$ length of subnormal at $(\sqrt{2}, -\sqrt{2})$

$= |(-\sqrt{2})(3)| = 3\sqrt{2}$

30 We have, $e^y = 1 + x^2$

$\Rightarrow \quad e^y\dfrac{dy}{dx} = 2x$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{2x}{1+x^2}$

$\Rightarrow \quad m = \dfrac{2x}{1+x^2},$

$|m| = \dfrac{2|x|}{|1+x^2|} = \dfrac{2|x|}{1+|x|^2} \le 1$

$\because \quad |x|^2 + 1 - 2|x| \ge 0 \ge$

$\Rightarrow \quad (|x| - 1)^2 \ge 0$

$\Rightarrow \quad |x|^2 + 1 \ge 2|x|$

$\Rightarrow \quad 1 \ge \dfrac{2|x|}{1+|x|^2}$

31 Clearly, the point of intersection of curves is (0, 1).

Now, slope of tangent of first curve,

$$m_1 = \frac{dy}{dx} = a^x \log a$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(0,1)} = m_1 = \log a$$

Slope of tangent of second curve,

$$m_2 = \frac{dy}{dx} = b^x \log b$$

$$\Rightarrow m_2 = \left(\frac{dy}{dx}\right)_{(0,1)} = \log b$$

$$\therefore \tan \alpha = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\log a - \log b}{1 + \log a \log b}$$

32 We have, $y^2 = 6x$

$$\Rightarrow 2y\frac{dy}{dx} = 6$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{y}$$

Slope of tangent at (x_1, y_1) is $m_1 = \frac{3}{y_1}$

Also, $9x^2 + by^2 = 16$

$$\Rightarrow 18x + 2by\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-9x}{by}$$

Slope of tangent at (x_1, y_1) is $m_2 = \frac{-9x_1}{by_1}$

Since, these are intersection at right angle.

$$\therefore m_1 m_2 = -1 \Rightarrow \frac{27x_1}{by_1^2} = 1$$

$$\Rightarrow \frac{27x_1}{6bx_1} = 1 \qquad [\because y_1^2 = 6x_1]$$

$$\Rightarrow b = \frac{9}{2}$$

33 $\because$ $y = x^2 - 5x + 6$

$$\therefore \frac{dy}{dx} = 2x - 5$$

Now, $m_1 = \left(\frac{dy}{dx}\right)_{(2,0)} = 4 - 5 = -1$

and $m_2 = \left(\frac{dy}{dx}\right)_{(3,0)} = 6 - 5 = 1$

Now, $m_1 m_2 = -1 \times 1 = -1$

Hence, angle between the tangents is $\frac{\pi}{2}$.

34 Slope of the curve, $\frac{x^2}{\alpha} + \frac{y^2}{4} = 1$ is

$m_1 = \frac{-4x}{\alpha y}$.

Now, slope of the curve, $y^3 = 16x$ is

$m_2 = \frac{16}{3y^2}$.

Now, apply the condition of perpendicularity of two curves,

i.e. $m_1 m_2 = -1$

and get $\alpha = \frac{4}{3}$ with the help of equation of curves.

35 $\int_1^2 f'(x)dx = [f(x)]_1^2 = f(2) - f(1) = 0$

[$\because f(x)$ satisfies the conditions of Rolle's theorem]

$\therefore \quad f(2) = f(1)$

36 $\because f(x) = x^3 - 6x^2 + ax + b$

On differentiating w.r.t. x, we get

$$f'(x) = 3x^2 - 12x + a$$

By the definition of Rolle's theorem

$$f'(c) = 0 \Rightarrow f'\left(2 + \frac{1}{\sqrt{3}}\right) = 0$$

$$\Rightarrow 3\left(2 + \frac{1}{\sqrt{3}}\right)^2 - 12\left(2 + \frac{1}{\sqrt{3}}\right) + a = 0$$

$$\Rightarrow 3\left(4 + \frac{1}{3} + \frac{4}{\sqrt{3}}\right) - 12\left(2 + \frac{1}{\sqrt{3}}\right) + a = 0$$

$$\Rightarrow 12 + 1 + 4\sqrt{3} - 24 - 4\sqrt{3} + a = 0$$

$$\Rightarrow a = 11$$

37 Since, $f(x)$ satisfies all the conditions of Rolle's theorem in [3, 5].

Let $f(x) = (x-3)(x-5) = x^2 - 8x + 15$

Now, $\int_3^5 f(x)dx = \int_3^5 (x^2 - 8x + 15)dx$

$$= \left[\frac{x^3}{3} - \frac{8x^2}{2} + 15x\right]_3^5$$

$$= \left(\frac{125}{3} - 100 + 75\right) - (9 - 36 + 45)$$

$$= \frac{50}{3} - 18 = -\frac{4}{3}$$

38 Using mean value theorem,

$$f'(c) = \frac{f(3) - f(1)}{3 - 1} \qquad \left[\because f'(c) = \frac{f(b) - f(a)}{b - a}\right]$$

$$\Rightarrow \frac{1}{c} = \frac{\log e\ 3 - \log e\ 1}{2}$$

$$\therefore c = \frac{2}{\log e\ 3} = 2\log_3 e$$

39 Given that, equation of curve

$y = x^3 = f(x)$

So, $f(2) = 8$ and $f(-2) = -8$

Now, $f'(x) = 3x^2 \Rightarrow f'(x) = \frac{f(2) - f(-2)}{2 - (-2)}$

$$\Rightarrow \frac{8 - (-8)}{4} = 3x^2$$

$$\therefore x = \pm\frac{2}{\sqrt{3}}$$

40 $f(x) = \sqrt{x}$

$\therefore f(a) = \sqrt{4} = 2$

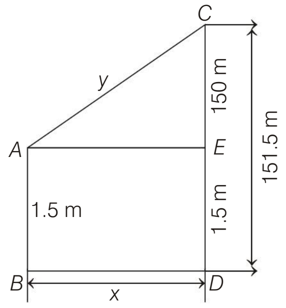
$f(b) = \sqrt{9} = 3;\ f'(x) = \frac{1}{2\sqrt{x}}$

Also, $f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{3 - 2}{9 - 4} = \frac{1}{5}$

$$\therefore \frac{1}{2\sqrt{c}} = \frac{1}{5} \Rightarrow C = \frac{25}{4} = 6.25$$

SESSION 2

1 Let AB be the position of boy who is flying the kite and C be the position of the kite at any time t.



Let $BD = x$ and $AC = y$, then $AE = x$

Given, $AB = 1.5$ m, $CD = 151.5$ m

$\therefore \quad CE = 150$ m

Given, $\frac{dx}{dt} = 10$ m/s

Here, we have to find $\frac{dy}{dt}$ when

$y = 250$ m

Now, from ΔCAE, $y^2 = x^2 + 150^2$

On differentiating, we get

$$2y\frac{dy}{dt} = 2x\frac{dx}{dt}$$

$$\Rightarrow \frac{dy}{dt} = \frac{x}{y} \cdot \frac{dx}{dt} = \frac{x}{y} \cdot 10 \qquad \text{...(i)}$$

In ΔACE, $x = \sqrt{250^2 - 150^2}$ $(\because y = 250)$

$= 200$ m

$\therefore$ From Eq. (i), we get

$$\frac{dy}{dt} = \frac{200}{250} \times 10 = 8 \text{ m/s}$$

2 Given curve is

$$y(x-2)(x-3) = x + 6 \qquad \text{...(i)}$$

Put $x = 0$ in Eq. (i), we get

$y(-2)(-3) = 6 \Rightarrow y = 1$

So, point of intersection is (0, 1).

Now, $y = \frac{x + 6}{(x-2)(x-3)}$

$$\Rightarrow \frac{dy}{dx} = \frac{1(x-2)(x-3) - (x+6)(x - 3 + x - 2)}{(x-2)^2(x-3)^2}$$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(0,1)} = \frac{6+30}{4 \times 9} = \frac{36}{36} = 1$

$\therefore$ Equation of normal at (0, 1) is given by

$y - 1 = \frac{-1}{1}(x - 0) \Rightarrow x + y - 1 = 0$

which passes through the point $\left(\frac{1}{2}, \frac{1}{2}\right)$.

3 Let $f(x) = (a+2)x^3 - 3ax^2 + 9ax - 1$ decreases monotonically for all $x \in R$, then $f'(x) \le 0$ for all $x \in R$

$\Rightarrow 3(a+2)x^2 - 6ax + 9a \le 0$ for all $x \in R$

$\Rightarrow (a+2)x^2 - 2ax + 3a \le 0$ for all $x \in R$

$\Rightarrow a + 2 < 0$ and discriminant ≤ 0

$\Rightarrow a < -2, -8a^2 - 24a \le 0$

$\Rightarrow a < -2$ and $a(a+3) \ge 0$

$\Rightarrow a < -2, a \le -3$ or $a \ge 0 \Rightarrow a \le -3$

$\therefore -\infty < a \le -3$

4 Now, $f'(x) = \frac{\sin x - x\cos x}{\sin^2 x}$

$= \frac{\cos x(\tan x - x)}{\sin^2 x}$

$\therefore f'(x) > 0$ for $0 < x \le 1$

So, $f(x)$ is an increasing function.

Now, $g'(x) = \frac{\tan x - x\sec^2 x}{\tan^2 x}$

$= \frac{\sin x\cos x - x}{\sin^2 x} = \frac{\sin 2x - 2x}{2\sin^2 x}$

Again, $\frac{d}{dx}(\sin 2x - 2x) = 2\cos 2x - 2$

$= 2(\cos 2x - 1) < 0$

So, $\sin 2x - 2x$ is decreasing.

$\Rightarrow \sin 2x - 2x < 0$

$\therefore g'(x) < 0$

So, $g(x)$ is decreasing.

5 $f'(x) = |\log_2[\log_3\{\log_4(\cos x + a)\}]|$

Clearly, $f(x)$ is increasing for all values of x, if

$\log_2[\log_3\{\log_4(\cos x + a)\}]$ is defined for all values of x.

$\Rightarrow \log_3[\log_4(\cos x + a)] > 0, \forall x \in R$

$\Rightarrow \log_4(\cos x + a) > 1, \forall x \in R$

$\Rightarrow \cos x + a > 4, \forall x \in R$

$\therefore a > 5$

6 Since, $\frac{f(2) - f(0)}{2 - 0} = f'(x)$

$\Rightarrow \frac{f(2) - 0}{2} = f'(x) \Rightarrow \frac{df(x)}{dx} = \frac{f(2)}{2}$

$\Rightarrow f(x) = \frac{f(2)}{2}x + C$

$\because f(0) = 0 \Rightarrow C = 0$

$\therefore f(x) = \frac{f(2)}{2}x$ ...(i)

Also, $|f'(x)| \le \frac{1}{2} \Rightarrow \left|\frac{f(2)}{2}\right| \le \frac{1}{2}$ ...(ii)

From Eq. (i), $|f(x)|$

$= \left|\frac{f(2)}{2}x\right| = \left|\frac{f(2)}{2}\right||x| \le \frac{1}{2}|x|$ [from Eq. (ii)]

In interval [0, 2], for maximum x,

$|f(x)| \le \frac{1}{2} \cdot 2 \Rightarrow |f(x)| \le 1$ [$\because x = 2$]

7 $g'(x) = f'(\sin x) \cdot \cos x - f'(\cos x) \cdot \sin x$

$\Rightarrow g''(x) = -f'(\sin x) \cdot \sin x + \cos^2 x f''(\sin x) + f''(\cos x) \cdot \sin^2 x - f'(\cos x) \cdot \cos x > 0, \forall x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow g'(x)$ is increasing in $\left(0, \frac{\pi}{2}\right)$.

Also, $g'\left(\frac{\pi}{4}\right) = 0$

$\Rightarrow g'(x) > 0, \forall x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

and $g'(x) < 0, \forall x \in \left(0, \frac{\pi}{4}\right)$

Thus, $g(x)$ is decreasing in $\left(0, \frac{\pi}{4}\right)$.

8 We have, $f(x) = (x-p)(x-q)(x-r)$

$\Rightarrow f(p) = 0 = f(q) = f(r)$

$\Rightarrow p, q$ and r are three distinct real roots of $f(x) = 0$

So, by Roolle's theorem, $f'(x)$ has one real root in the interval (p, q) and other in the interval (q, r). Thus, $f'(x)$ has two distinct real roots.

Now, $f(x) = (x-p)(x-q)(x-r)$

$\Rightarrow f(x) = x^3 - x^2(p+q+r) + x(pq+qr+rp) - pqr$

$\Rightarrow f'(x) = 3x^2 - 2(p+q+r)x + (pq+qr+rp)$

As $f'(x)$ has distinct real roots

$\therefore 4(p+q+r)^2 - 12(pq+qr+rp) > 0$

$\Rightarrow (p+q+r)^2 > 3(pq+qr+rp)$

9 Given that, $f(x)$ is monotonic.

$\Rightarrow f'(x) = 0$ or $f'(x) > 0, \forall x \in R$

$\Rightarrow f'(px) < 0$ or $f'(px) > 0, \forall x \in R$

So, $f'(px)$ is also monotonic.

Hence, $f(x) + f(3x) + \ldots + f[(2m-1)x]$ is a monotonic.

Polynomial of odd degree $(2m - 1)$, so it will attain all real values only once.

10 Since, the balloon is spherical in shape, hence the volume of the balloon is

$V = \frac{4}{3}\pi r^3$.

On differentiating both the sides w.r.t. t, we get

$\frac{dV}{dt} = \frac{4}{3}\pi\left(3r^2 \times \frac{dr}{dt}\right)$

$\Rightarrow \frac{dr}{dt} = \frac{dV/dt}{4\pi r^2}$ ...(i)

Now, to find $\frac{dr}{dt}$ at the rate $t = 49$ min, we require $\frac{dV}{dt}$ the radius (r) at that stage. $\frac{dV}{dt} = -72\pi \text{m}^3/\text{min}$

Also, amount of volume lost in 49 min $= 72\pi \times 49 \text{m}^3$

$\therefore$ Final volume at the end of 49 min $= (4500\pi - 3528\pi)\text{m}^3$

$= 972\pi \text{m}^3$

If r is the radius at the end of 49 min, then $\frac{4}{3}\pi r^3 = 972\pi \Rightarrow r^3 = 729$

$\Rightarrow r = 9$

Radius of the balloon at the end of 49 min = 9 m

From Eq. (i),

$\frac{dr}{dt} = \frac{dV/dt}{4\pi r^2} \Rightarrow \left(\frac{dr}{dt}\right)_{t=49} = \frac{\left(\frac{dV}{dt}\right)_{t=49}}{4\pi(r^2)_{t=49}}$

$\left(\frac{dr}{dt}\right)_{t=49} = \frac{72\pi}{4\pi(9^2)} = \frac{2}{9}$ m/min

11 Given equation of curve is

$x^2 + 2xy - 3y^2 = 0$

On differentiating w.r.t. x, we get

$2x + 2xy' + 2y - 6yy' = 0$

$\Rightarrow y' = \frac{x+y}{3y-x}$

At $x = 1, y = 1, y' = 1$ i.e. $\left(\frac{dy}{dx}\right)_{(1,1)} = 1$

Equation of normal at (1, 1) is

$y - 1 = -\frac{1}{1}(x-1) \Rightarrow y - 1 = -(x-1)$

$\Rightarrow x + y = 2$

On solving Eqs. (i) and (ii) simultaneously, we get

$x^2 + 2x(2-x) - 3(2-x)^2 = 0$

$\Rightarrow x^2 + 4x - 2x^2 - 3(4 + x^2 - 4x) = 0$

$\Rightarrow -x^2 + 4x - 12 - 3x^2 + 12x = 0$

$\Rightarrow -4x^2 + 16x - 12 = 0$

$\Rightarrow 4x^2 - 16x + 12 = 0$

$\Rightarrow x^2 - 4x + 3 = 0$

$\Rightarrow (x-1)(x-3) = 0$

$\Rightarrow x = 1, 3$

Now, when $x = 1$, then $y = 1$

and when $x = 3$, then $y = -1$

$\therefore P = (1, 1)$ and $Q = (3, -1)$

Hence, normal meets the curve again at $(3, -1)$ in fourth quadrant.

Alternate Method

Given, $x^2 + 2xy - 3y^2 = 0$

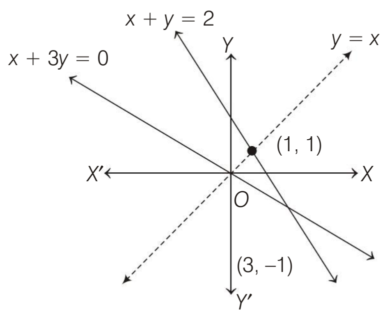
$\Rightarrow (x-y)(x+3y) = 0$

$\Rightarrow x - y = 0$ or $x + 3y = 0$

Equation of normal at (1, 1)

$y - 1 = -1(x-1) \Rightarrow x + y - 2 = 0$

It intersects $x + 3y = 0$ at $(3, -1)$ and hence normal meets the curve in fourth quadrant.



12 Given, $f(0) = 2 = g(1), g(0) = 0$

and $\quad f(1) = 6$

f and g are differentiable in $(0,1)$.

Let $\quad h(x) = f(x) - 2g(x) \qquad \ldots(i)$

$\Rightarrow \quad h(0) = f(0) - 2g(0)$

$\Rightarrow \quad h(0) = 2 - 0 \Rightarrow h(0) = 2$

and $\quad h(1) = f(1) - 2g(1) = 6 - 2(2)$

$\Rightarrow \quad h(1) = 2, h(0) = h(1) = 2$

Hence, using Rolle's theorem, we get

$h'(c) = 0$, such that $c \in (0,1)$

On differentiating Eq.(i) at c, we get

$f'(c) - 2g'(c) = 0 \Rightarrow f'(c) = 2g'(c)$

13 Let $y = ax^2 + bx + c$

[equation of parabola]

As it touches $y = x$ at $x = 1$.

$\therefore \quad y = a + b + c$

and $\quad y = 1 \Rightarrow a + b + c = 1$

Now, $\dfrac{dy}{dx} = 2ax + b$

$\Rightarrow \left(\dfrac{dy}{dx}\right)_{\text{at } x=1} = 2a + b \Rightarrow 2a + b = 1$

[from $y = x$, slope $= 1$]

Now, $f(x) = ax^2 + bx + c$

$\Rightarrow \quad f'(x) = 2ax + b \Rightarrow f''(x) = 2a$

$\therefore \quad f(0) = c, f'(0) = b, f''(0) = 2a,$

$f'(1) = 2a + b = 1$

14 $a + b = 4 \Rightarrow b = 4 - a$

and $b - a = 4 - 2a = t \qquad$ (say)

Now, $\int_0^a g(x)\,dx + \int_0^b g(x)\,dx = \int_0^a g(x)\,dx + \int_0^{4-a} g(x)\,dx = I(a)$

$\Rightarrow \quad \dfrac{dI(a)}{da} = g(a) - g(4-a)$

As $a < 2$ and $g(x)$ is increasing.

$\Rightarrow 4 - a > a \Rightarrow g(a) - g(4-a) < 0$

$\Rightarrow \quad \dfrac{dI(a)}{da} < 0$

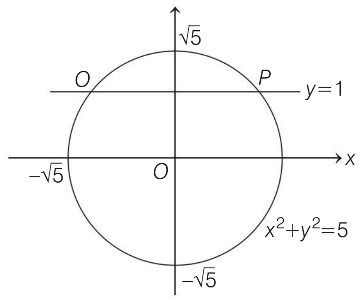
Now, $\dfrac{dI(a)}{d(a)} = \dfrac{dI(a)}{dt}\dfrac{dt}{da} = -2 \cdot \dfrac{dI(a)}{dt}$

$\Rightarrow \quad \dfrac{dI(a)}{dt} > 0$

Thus, $I(a)$ is an increasing function of t. Hence, the given expression increasing with $(b - a)$.

15 We know that,

$1 \le |\sin x| + |\cos x| \le \sqrt{2}$



$\Rightarrow \quad y = [|\sin x| + |\cos x|] = 1$

Let P and Q be the points of intersection of given curves.

Clearly, the given curves meet at points where $y = 1$, so we get

$x^2 + 1 = 5 \Rightarrow x = \pm 2$

Now, $\quad P(2, 1)$ and $Q(-2, 1)$

On differentiating $x^2 + y^2 = 5$ w.r.t. x, we get

$2x + 2y\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{x}{y}$

$\left(\dfrac{dy}{dx}\right)_{(2,1)} = -2$

and $\left(\dfrac{dy}{dx}\right)_{(-2,1)} = 2$

Clearly, the slope of line $y = 1$ is zero and the slope of the tangents at P and Q are (-2) and (2), respectively.

Thus, the angle of intersection is $\tan^{-1}(2)$.

16 There is only one function in option (a), whose critical point $\frac{1}{2} \in (0,1)$ but in other parts critical point $0 \notin (0,1)$. Then, we can say that functions in options (b), (c) and (d) are continuous on [0, 1] and differentiable in (0, 1).

Now, for $f(x) = \begin{cases} \frac{1}{2} - x, & x < \frac{1}{2} \\ \left(\frac{1}{2} - x\right)^2, & x \ge \frac{1}{2} \end{cases}$

Here, $Lf'\left(\dfrac{1}{2}\right) = -1$

and $Rf'\left(\dfrac{1}{2}\right) = 2\left(\dfrac{1}{2} - \dfrac{1}{2}\right)(-1) = 0$

$\therefore \quad Lf'\left(\dfrac{1}{2}\right) \ne Rf'\left(\dfrac{1}{2}\right)$

$\Rightarrow f$ is non-differentiable at $x = \dfrac{1}{2} \in (0,1)$.

$\therefore$ Lagrange mean value theorem is not applicable to $f(x)$ in [0, 1].